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Sign-Change Diminishing Systems of Functions of Many Variables

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Given functions $f_1(\mathbf{x}), ..., f_n(\mathbf{x})$, $\mathbf{x} = (x_1, ..., x_m) \in M$, where M is an open parallelepiped or simplex, let all minors of the matrix

$$\left(\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right)_{i=1,...,n}^{j=1,...,m}$$

be positive for all $\mathbf{x} \in M$. It is shown that if the sequence $y_1 - x_1, ..., y_m - x_m$ with $(x_1, ..., x_m), (y_1, ..., y_m) \in M$ has k sign-changes, then there are no more than k sign changes in the sequence $f_1(\mathbf{y}) - f_1(\mathbf{x}), ..., f_n(\mathbf{y}) - f_n(\mathbf{x})$.

1. THE MAIN RESULT

We first introduce the symbols $\prec_k$ and $\prec^k$, where k is a natural number. Let y_i, z_i, $i = 1, ..., r$, be real numbers. We write

$$(y_1, ..., y_r) \prec_k (z_1, ..., z_r)((y_1, ..., y_r) \prec^k (z_1, ..., z_r)), \tag{1}$$

iff there do not exist integers $1 \leqslant s_1 \leqslant \cdots < s_k \leqslant r$ such that

$$(-1)^{k-i} y_{s_i} > (-1)^{k-i} z_{s_i}((-1)^{k-i} y_{s_i} \geqslant (-1)^{k-1} z_{s_i}), \qquad i = 1, ..., k.$$

The relation (1) means that there are no more than $k-1$ sign-changes in the sequence $z_1 - y_1, ..., z_r - y_r$. Furthermore, if the number of sign-changes is $k-1$ then the last sign is $+$. The difference between $\prec_k$ and $\prec^k$ is in the way of counting sign-changes. In the first case we omit zeros; in the second one we assign to them any sign.

DEFINITION 1. A system of functions

$$f_1(\mathbf{x}) = f_1(x_1, ..., x_m), ..., f_n(\mathbf{x}) = f_n(x_1, ..., x_m), \tag{2}$$

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defined on $M \subset R^m$ is called sign-change diminishing iff for any $\mathbf{y}, \mathbf{z} \in M$, $\mathbf{y} \neq \mathbf{z}$, and integer $k \geqslant 1$ the relation $\mathbf{y} \prec_k \mathbf{z}$ implies

$$(f_1(\mathbf{y}), ..., f_n(\mathbf{y})) \prec^k (f_1(\mathbf{z}), ..., f_n(\mathbf{z})).$$

It is known [1, 2] that the system (2) with

$$f_i(\mathbf{x}) = a_{i1}x_1 + \cdots + a_{im}x_m + b_i, \qquad i = 1, ..., n,$$

$M = R^m$ is sign-change diminishing iff the matrix $(a_{ij})_{i=1}^{n}\,{}_{j=1}^{m}$ is strictly totally positive, viz. all its minors are positive. As far as we are aware, the nonlinear case has never been considered in the literature. If the functions of (2) are continuously differentiable, it is natural to require the strict total positivity of the matrices

$$\left(\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right)_{i=1,...,n}^{j=1,...,m} \tag{3}$$

for all $x \in M$. But, generally speaking, this does not mean that the system (2) is sign-change diminishing.

EXAMPLE 1. Let

$$f_1(x_1, x_2) = x_1 + x_2, \qquad f_2(x_1, x_2) = x_1 - \frac{x_1^3}{3} + \frac{x_2^3}{3} \tag{4}$$

be defined on $M = \{(x_1, x_2) : -1 < x_1 < 1,\ x_2 > 0,\ x_1^2 + x_2^2 > 1\}$.

In this example the matrix (3) is

$$\begin{pmatrix} 1 & 1 \\ 1 - x_1^2 & x_2^2 \end{pmatrix}. \tag{5}$$

We have $1 - x_1^2 > 0$, $x_2^2 > 0$, $x_2^2 - (1 - x_1^2) > 0$ for all $(x_1, x_2) \in M$ and therefore the matrix (5) is strictly totally positive on M. Since $f_1(1, 0) = 1$, $f_2(1, 0) = 2/3$, we can choose a point $(x_1', x_2') \in M$ sufficiently close to $(1, 0)$ such that

$$x_1' > 0, \qquad x_2' < 1.1, \tag{6}$$

$$f_1(x_1', x_2') = x_1' + x_2' < 1.1 = f_1(0, 1.1), \tag{7}$$

$$f_2(x_1', x_2') > \frac{(1.1)^3}{3} = f_2(0, 1.1). \tag{8}$$

The inequalities (6) imply the relation $(x'_1, x'_2) \prec_2 (0, 1.1)$. From (7) and (8) we conclude that the relation

$$(f_1(x'_1, x'_2), f_2(x'_1, x'_2)) \prec^2 (f_1(0, 1.1), f_2(0, 1.1))$$

does not hold. Thus the system of functions (4) is not sign-change diminishing in spite of the strict total positivity of the matrix (3) for all $\mathbf{x} \in M$.

In this paper we prove that if M is an open parallelepiped or simplex and the matrix (3) is strictly totally positive for all $\mathbf{x} \in M$, then the system (2) is sign-change diminishing.

Let (2) be a system of continuously differentiable functions defined on one of the sets

$$\text{(i)} \quad M = \{(x_1, ..., x_m) : -\infty \leqslant a_i < x_i < b_i \leqslant +\infty, i = 1, ..., m\},$$

$$\text{(ii)} \quad M = \{(x_1, ..., x_m) : -\infty \leqslant a < x_1 < \cdots < x_m < b \leqslant +\infty\}, \text{ or}$$

$$\text{(iii)} \quad M = \{(x_1, ..., x_m) : +\infty \geqslant b > x_1 > \cdots > x_m > a \geqslant -\infty\}.$$

THEOREM 1. *If all minors of the matrix* (3) *are positive for all* $x \in M$, *and* M *is as above in* (i), (ii), *or* (iii), *then the system of continuously differentiable functions* (2) *is sign-change diminishing.*

2. Proof

Let k be the minimal natural number for which there exist $\mathbf{x}' = (x'_1, ..., x'_m) \in M$, $\mathbf{x}'' = (x''_1, ..., x''_m) \in M$ such that $\mathbf{x}' \prec_k \mathbf{x}''$ but the relation $\mathbf{f}(\mathbf{x}') \prec^k \mathbf{f}(\mathbf{x}'')$ does not hold.

M as in (i). We first consider $k = 1$. The relation $\mathbf{x}' \prec_1 \mathbf{x}''$ implies $x'_j \leqslant x''_j$ for $j = 1, ..., m$. Since the matrix (3) is strictly totally positive, we have $\partial f_i(\mathbf{x})/\partial x_j > 0$ for all $i = 1, ..., n$, $j = 1, ..., m$, and $\mathbf{x} \in M$. Thus $\mathbf{x}' \prec_1 \mathbf{x}''$ implies the inequalities

$$f_i(x'_1, ..., x'_m) < f_i(x''_1, ..., x''_m), \qquad i = 1, ..., n.$$

These inequalities may be rewritten as $\mathbf{f}(\mathbf{x}') \prec^1 \mathbf{f}(\mathbf{x}'')$. This proves the result for $k = 1$.

Now let $k > 1$. Using the notation $x_j^* = \max\{x'_j, x''_j\}$ and $x_{*j} = \min\{x'_j, x''_j\}$ for $j = 1, ..., m$, we consider the subset

$$M_1 = \{\mathbf{x} = (x_1, ..., x_m) : x_{*j} \leqslant x_j \leqslant x_j^*, j = 1, ..., m\}$$

of the parallelepiped M.

The relation $\mathbf{f}(\mathbf{x}') \prec^k \mathbf{f}(\mathbf{x}'')$ does not hold. This means that there are integers $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that

$$(-1)^{k-r} f_{i_r}(\mathbf{x}') \geqslant (-1)^{k-r} f_{i_r}(\mathbf{x}''), \qquad r = 1, ..., k.$$

Since the functions (2) are continuous, there is $\mathbf{x}^o = (x_1^o, ..., x_m^o)$ from $M_1' = \{\mathbf{x} \in M_1 \subset M : f_{i_r}(\mathbf{x}) = f_{i_r}(\mathbf{x}'), r = 1, ..., k-1\}$ such that $f_{i_k}(\mathbf{x}^o) \geqslant f_{i_k}(\mathbf{x})$ for all $\mathbf{x} \in M_1'$. We can write $\mathbf{x}^o \neq \mathbf{x}''$ because if $f_{i_k}(\mathbf{x}') = f_{i_k}(\mathbf{x}'') = f_{i_k}(\mathbf{x}^o)$ we define $\mathbf{x}^o = \mathbf{x}'$. From the definition of $\mathbf{x}^o$, the relation $\mathbf{f}(\mathbf{x}^o) \prec^k \mathbf{f}(\mathbf{x}'')$ also does not hold. Hence, by the definition of k neither relation $\mathbf{x}^o \prec_{k-1} \mathbf{x}''$ nor $\mathbf{x}'' \prec_{k-1} \mathbf{x}^o$ is valid. Since $\mathbf{x}^o \in M_1$ we have $\mathbf{x}^o \prec_k \mathbf{x}''$. Consequently, there exist integers $1 \leqslant j_1 < \cdots < j_k \leqslant m$ such that

$$(-1)^{k-s} x_{j_s}^o < (-1)^{k-s} x_{j_s}'', \qquad s = 1, ..., k.$$

Therefore

$$(-1)^{k-s} x_{j_s}' \leqslant (-1)^{k-s} x_{j_s}^o < (-1)^{k-s} x_{j_s}'', \qquad s = 1, ..., k. \tag{9}$$

Let us consider the functions $f_{i_r}(\mathbf{x})$, $r = 1, ..., k$, as functions of the k variables $x_{j_1}, ..., x_{j_k}$ with the other variables $x_i^o = x_i$, $i \neq j_1, ..., j_k$, fixed, as defined on the domain $a_{j_1} < x_{j_1} < b_{j_1}, ..., a_{j_k} < x_{j_k} < b_{j_k}$. The matrix (3) is strictly totally positive. Hence

$$A = \left| \frac{\partial f_{i_r}^o(\mathbf{x})}{\partial x_{j_s}} \right| > 0.$$

Thus, the system of equations $y_{i_1} = f_{i_1}(\mathbf{x}), ..., y_{i_k} = f_{i_k}(\mathbf{x})$ satisfies the conditions of the inverse function theorem (see [3]). Therefore:

1. There exists a one-to-one correspondence between sufficiently small neighbourhoods X and Y of the points $(x_{j_1}^o, ..., x_{j_k}^o)$ and $(y_{i_1}^o, ..., y_{i_k}^o)$, respectively, where $y_{i_r}^o = f_{i_r}(\mathbf{x}^o)$, $r = 1, ..., k$.

In particular, for ε small and points $(y_{i_1}^o, ..., y_{i_k}^o + \varepsilon) \in Y$ there exists a unique point $(x_{j_1}(\varepsilon), ..., x_{j_k}(\varepsilon)) \in X$ such that

$$y_{i_r}^o = f_{i_r}(\mathbf{x}(\varepsilon)), \qquad r = 1, ..., k-1, \qquad y_{i_k}^o + \varepsilon = f_{i_k}(\mathbf{x}(\varepsilon)), \tag{10}$$

where $\mathbf{x}(\varepsilon) = (x_1(\varepsilon), ..., x_m(\varepsilon))$, $x_i(\varepsilon) = x_i^o$ for $i \neq j_1, ..., j_k$.

2. The inverse equations $x_{j_s} = x_{j_s}(y_{i_1}, ..., y_{i_k})$, $s = 1, ..., k$, are continuously differentiable at the point $(y_{i_1}^o, ..., y_{i_k}^o)$. Hence, there are derivatives

$$\dot{x}_{j_s}(0) = \dot{x}_{j_s}(\varepsilon)_{|\varepsilon=0} = \frac{\partial x_{j_s}(y_{i_1}^o, ..., y_{i_k}^o)}{\partial y_{i_k}}, \qquad s = 1, ..., k.$$

Differentiating the equalities (10) with respect to ε at the point $\varepsilon = 0$, we obtain

$$\left(\frac{\partial f_{i_r}(\mathbf{x}^o)}{\partial x_{j_s}}\right)_{r,s=1}^{k} \begin{pmatrix} \dot{x}_{j_1}(0) \\ \cdots\cdots \\ \dot{x}_{j_k}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}. \tag{11}$$

By Cramer's rule we have

$$\dot{x}_{j_s}(0) = (-1)^{k-s} \frac{A_s}{A}, \qquad s = 1, ..., k,$$

where A_s is the determinant of the matrix which we obtain from the matrix in (11) by omitting the nth row and the sth column.

Since $A > 0$ and $A_s > 0$ for all $s = 1, ..., k$, we conclude that

$$(-1)^{k-s} \dot{x}_{j_s}(0) > 0, \qquad s = 1, ..., k. \tag{12}$$

By definition $x_j(0) = x_j^o$ for all $j = 1, ..., m$. Therefore, by (12) and (9) there exists a $\delta > 0$ such that

$$(-1)^{k-s} x'_{j_s} \leqslant (-1)^{k-s} x^o_{j_s} < (-1)^{k-s} x_{j_s}(\delta) < (-1)^{k-s} x''_{j_s}, \qquad s = 1, ..., k.$$

Thus, $\mathbf{x}(\delta) \in M'_1$ and by (10) we have $f_{i_k}(\mathbf{x}(\delta)) = f_{i_k}(\mathbf{x}^o) + \delta$. This is a contradiction to the definition of $\mathbf{x}^o$.

M as in (ii). We use the notation:

$$\Delta(\mathbf{x}) = \min\{x_2 - x_1, ..., x_m - x_{m-1}\}, \qquad \mathbf{x} = (x_1, ..., x_m) \in M;$$

$$M_o = \{\mathbf{x} \in M : \Delta(\mathbf{x}) \geqslant \Delta'\} \qquad \text{with} \quad 0 < \Delta' < \min\{\Delta(\mathbf{x}'), \Delta(\mathbf{x}'')\};$$

$$M_1 = \{\mathbf{x} \in M_o : x_{*j} \leqslant x_j \leqslant x_j^*, 1 \leqslant j \leqslant m\},$$

where $x_{*j} = \min\{x'_j, x''_j\}$, $x_j^* = \max\{x'_j, x''_j\}$, $j = 1, ..., m$.

Since the relation $\mathbf{f}(\mathbf{x}') \prec^k \mathbf{f}(\mathbf{x}'')$ does not hold, there are integers $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that

$$(-1)^{k-r} f_{i_r}(\mathbf{x}') \geqslant (-1)^{k-r} f_{i_r}(\mathbf{x}''), \qquad r = 1, ..., k.$$

The functions (2) are continuous and therefore there exists $\mathbf{x}^o = (x_1^o, ..., x_m^o)$ from $M'_1 = \{\mathbf{x} \in M_1 : f_{i_r}(\mathbf{x}) = f_{i_r}(\mathbf{x}'), r = 1, ..., k-1\}$ such that $f_{i_k}(\mathbf{x}^o) \geqslant f_{i_k}(\mathbf{x})$ for all $\mathbf{x} \in M'_1$. We can write $\mathbf{x}^o \neq \mathbf{x}''$, because if $f_{i_k}(\mathbf{x}') = f_{i_k}(\mathbf{x}'') = f_{i_k}(\mathbf{x}^o)$, then we set $\mathbf{x}^o = \mathbf{x}'$. According to the definition of $\mathbf{x}^o$, the relation $\mathbf{f}(\mathbf{x}^o) \prec^k \mathbf{f}(\mathbf{x}'')$ also does not hold. Hence, by the definition of k, neither relation $\mathbf{x}^o \prec_{k-1} \mathbf{x}''$ nor $\mathbf{x}'' \prec_{k-1} \mathbf{x}^o$ is valid. Since $\mathbf{x}^o \in M_1$, we have

$\mathbf{x}^o \prec_k \mathbf{x}''$. Therefore, there exist integers $1 \leqslant j_1 < \cdots < j_k \leqslant m$ such that $(-1)^{n-s} x_{j_s}^o < (-1)^{n-s} x_{j_s}''$, $s = 1, ..., n$. We suppose that the integers are chosen so that the sum $\sum_{s=1}^{n} (-1)^{n-s} j_s$ is maximal. Then $x_{j_s}^o < x_{j_s+1}^o - \Delta'$ if $n-s$ is even and $j_s < m$. Indeed, otherwise $x_{j_s+1}^o = x_{j_s}^o + \Delta'$; hence $x_{j_s+1}^o < x_{j_s+1}''$ and the sum (12) is not maximal. Analogously, $x_{j_s-1}^o + \Delta' < x_{j_s}^o$ if $n-s$ is odd and $j_s > 1$.

Examining the functions $f_{i_r}(\mathbf{x})$, $r = 1, ..., k$, as functions of the k variables $x_{j_1}, ..., x_{j_k}$ with the other variables $x_i = x_i^o$, $i \neq j_1, ..., j_k$, fixed, the proof can now proceed as does the proof in the previous case. We note that the number $\delta > 0$ must also obey the inequalities

$$x_{j_s}(\delta) < x_{j_s+1}^o - \Delta' \qquad \text{when} \quad n-s \text{ is even and } j_s < m,$$

$$x_{j_s-1} + \Delta' < x_{j_s}(\delta) \qquad \text{when} \quad n-s \text{ is odd and } j_s > 1.$$

The proof in the case where M is as in (iii) is similar to that for the previous one.

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